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Thesis

Proofs of the Fundamental Theorem

of

Algebra

Submitted by

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Outline of

A Presentation of Proofs of the

Fundamental Theorem of Algebra.

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I.

THEOREM I. Given $\phi(z) = 1 + bz^m + cz^{m+1} + - - + kz^n$ where b, c, - - - k, denote constants, real or complex and z a complex variable; it is always possible to choose z so that $|\phi(z)| < 1$.

Let the expression for z and b in terms of absolute value and amplitude be

$$z = \rho(\cos\theta + i\sin\theta) \qquad b = |b| \cdot (\cos\beta + i\sin\beta).$$

then $bz^m = \rho^m \cdot |b| [\cos(m\theta + \beta) + i\sin(m\theta + \beta)$

Choosing θ so that $m\theta + \beta = \pi$

then $bz^m = \rho^m |b| (\cos\pi + i\sin\pi)$ or

$= -\rho^m |b|$

Now choose ρ so that

$$|c|\rho^{m+1} + - - - + |k|\rho^n < |b|\rho^m < 1$$

If z_0 denote the value of z corresponding to these values of θ and ρ then

$|\phi(z_0)| < 1$ for since

$$\phi(z_0) = (1 - \rho^m |b|) + cz_0^{m+1} + - - - - + kz_0^n$$

$$|\phi(z_0)| \leqq 1 - \rho^m |b| + |c|\rho^{m+1} + - - - - |k|\rho^n \therefore < 1$$

(the sum of the absolute values of two complex numbers can not be less than the absolute value of the sum)

Given: the function $f(z) = a_0z^n + a_1z^{n-1} + - - - + a_n$;

If f(z) does not vanish when $z = b$, we can always choose z so that $|f(z)| < |f(b)|$.

For in f(z) place $z = b + h$ and develop by Taylor's Theorem.

It may happen that some of the derivatives $f'(z)$, $f''(z)$, etc. may vanish when $z = b$; but, they can not all vanish since $f^n(z) = n!a_0$.

Let $f^m(z)$ denote the first one which does not vanish.

Then $f(b + h) = f(b) + f^m(b)\dfrac{h^m}{m!} + - - - - + f^n(b)\dfrac{h^n}{n!}$

and $\therefore \dfrac{f(b+h)}{f(b)} = 1 + \dfrac{f^m(b)}{f(b)}\dfrac{h^m}{m!} + - - - - + \dfrac{f^n(b)}{f(b)}\dfrac{h^n}{n!}$

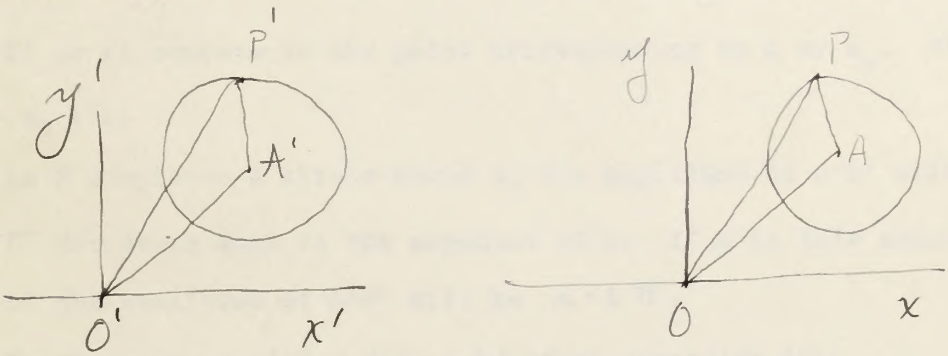
The second member here is of the form of our polynomial in the first paragraph. Hence we can choose h so that

$$\frac{|f(b+h)|}{|f(b)|} < 1 \text{ and hence that } |f(b+h)| < f|(b)| .$$

II.

CONSIDERATION OF CONTINUOUS FUNCTIONS. f(z) represents a rational integral function of the n^{th} degree and is obviously continuous.

Representing continuous series of values as points in two separate planes (or even two sets of points on the same plane) we get a curve representing z in one and f(z) in the other.



Since f(z) is a rational integral function and therefore analytic, if P' corresponds to value of f(z) when z has the value corresponding to P, then as P is allowed to describe a closed curve, P' will follow a curve. When P returns to its original position after following its curve P' must return to its position and hence follows a closed curve. The curve followed by P' may not be as simple as shown above. It may cross itself in returning to the original position. Our simple diagram is sufficient to bring out the result.

Let us consider what happens to the amplitude if A be a determined point (x_0, y_0) where $z_0 = x_0 + iy_0$.

There are two cases to be considered. (1) Where z_0 is not a root of f(z) and (2) where it is.

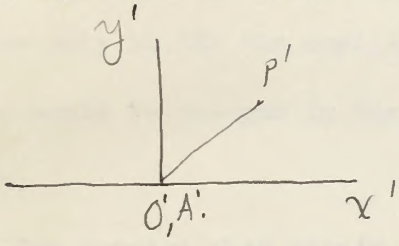
(1) In the first case suppose $z = z_0 + h$ where $|z_0 + h| <$ OA and also

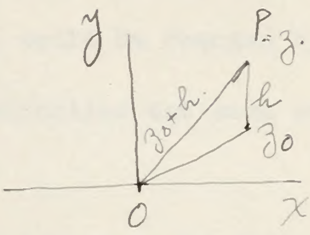
where the corresponding $|f(z)| < O'A'$. This is possible since A' corresponding to A is not O'. It is evident then that a complete description of its curve by P makes P' describe its complete curve and the total net change of amplitude in each is zero.

(2) In this case O'A' is zero and as P describes its closed curve P' also describes its closed curve but the amplitude of P' has increased by 2π.

So far we have considered z_0 as a simple root of the function f(z).

Observe what happens in the diagram as P follows its closed curve when z_0 is a multiple root of f(z) = 0.





O' or A' represents the point corresponding to A or z_0. P' corresponds to $P = z = z_0 + h$.

As P completes a circle about z_0 the amplitude of O'P' will have an increment of 2π for every unit in the exponent of h. If m is this exponent then the increment of the amplitude of O'P' will be $m \cdot 2\pi$.

We can express $f(z) = f(z_0 + h)$ as an expansion like

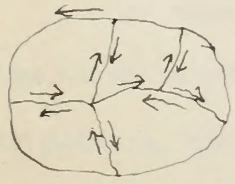
$f(z_0) + f'(z_0)h + \frac{f''(z_0)}{1\cdot 2} h^2 + - - + a_0 h^n$. Since $f(z_0)$ is 0, h is a factor which can be removed. By a succession of applications we can show $f(z) = h^m \psi(z)$ where $\psi(z)$ does not contain z_0 as a root.

The amp $f(z) = m\,\theta$ + amp $\psi(z)$

By the first case the amp of $\psi(z)$ is nothing.

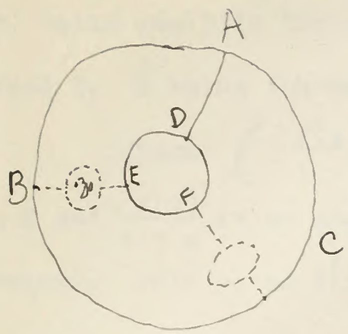
Since the increment of θ is 2π in one revolution, the increment of m is $m\,2\pi$. $\therefore$ the increment of amplitude of f(z) is $2\pi m$.

If a plane area in the z plane be divided into parts the variation of amplitude of f(z) corresponding to the description in the same sense by z of all the



partial areas is equal to the variation of amplitude of f(z) corresponding to the description of z of the external perimeter only.

In the case of a ring this is shown as follows:



Taking the ring ABC - DEF. Drawing line AD makes the region of the ring simply connected. Starting at A and keeping the area always on the left, a description of arc ABCA makes variation of amplitude of $z = 0$. Variation of $f(z)$ is zero or a multiple of 2π according to whether it has no roots as z follows ABC or has roots. Description of arc ABCA - AD - DFED - DA also gives complete variation of zero for z. If roots of $f(z)$ are only in DEF the amplitude of $f(z)$ would be changed by a description of ABC but it would be changed in the opposite direction and same amount by description of arc DFE.

Now suppose that points in a region are roots of $f(z)$. Enclose each root in a small closed curve (such as indicated by small circle about z_0 in the figure). The variation of amplitude of $f(z)$ corresponding to description of the boundary of any of the small curves containing a root gives a corresponding increment in $f(z)$ of $2 n_i \pi$ (n_i representing the order of any one of the roots). Then description of ABCDFE will cause variation of amplitude of $f(z)$ $\sum_1^m 2 n_i \pi$, m being the number of small curves.

Description of any of the other areas (not containing roots of $f(z)$ causes no variation of amplitude of $f(z)$.

Variation of amplitude of $f(z)$ for the external perimeter only then gives variation of amplitude of $f(z) = \sum_1^m 2 n_i \pi$.

III.

An analytic function $f(z)$ throughout a region T can not have a greater value at an interior point of the region than its greatest value on a circle about this point z, the circle and its boundary being interior to the region.

Consider Cauchy's Integral formula that $f(z) = \frac{1}{2\pi i} \int_C \frac{f(t)}{t-z} dt$

f(z) being analytic throughout open region T and single valued and continuous in closed T. C being the entire boundary.

Since $\int_C \frac{f(t)}{t-z} dt$ is a single valued and continuous function of z throughout T and $\frac{f(t)}{t-z}$ is an analytic function of z, it is possible to differentiate the integral. This gives $f'(z) = \frac{1}{2\pi i} \int_C \frac{f(t)}{(t-z)^2} dt$ which is again analytic in open T. since it has derivatives at every point of T.

This new integral has successive derivatives. The n^{th} one is

$$f^n(z) = \frac{n!}{2\pi i} \int_C \frac{f(t)}{(t-z)^{n+1}} dt \quad (1)$$

If no value on C of a continuous real function $|f(z)|$ is greater than a positive constant M, and the length of C is l, then $\int_C |f(z)| dz$ is not greater than the integral in which M replaces the integrand or

$$\left| \int_C f(z)\, dz \right| \leqq Ml.$$

From this since the absolute value of the integrand in (1) above is less than $\frac{M}{r^{n+1}}$ for all values of t on C we have

$$|f^n(z)| \leqq \frac{n!}{2\pi} \frac{M}{r^{n+1}} 2\pi r = (n!) \frac{M}{r^n}$$

$n! = 1$ if $n = 0$ which gives

$f^0(z) \leqq \frac{M}{r^0} = M$ which proves the theorem.

IV.

LIOUVILLE'S THEOREM. The above property makes it possible to prove a theorem of Liouville's stating that no function (except a constant) of a complex variable can be analytic everywhere and be finite.

In Cauchy's inequality if n is 1, $f'(z) \leqq \frac{M}{r}$ and so f'(z) can be made less than any assigned quantity by taking r as large as we please.

M may be fixed at a definite value here since we have assumed that f(z) is everywhere finite.

Since $f'(z) \leqq$ any assigned value when z is fixed, $f'(z) = 0$ for all values of z.

One value of $\int f'(z)dz = f(z)$.

Every indefinite integral of $f'(z)$ has the form $f(z) +$ constant.

$\therefore$ $f(z) +$ constant $= 0$

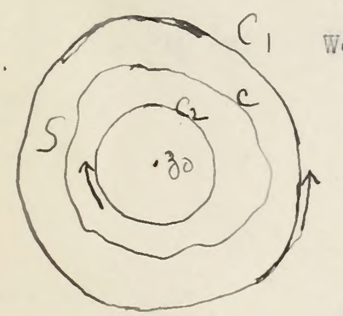
hence $f(z) =$ a constant.

V.

DEVELOPMENT OF LAURENT'S SERIES. In the neighborhood of a regular point of an analytic function. can be represented by a power series, but this method does not hold in the neighborhood of a singular point of the function. It will be shown that in the neighborhood of an isolated singular point z_0 we can expand an analytic function in a series which has some negative powers of $(z - z_0)$.

Taylor's theorem applies within a region bounded by a single curve (circle) provided there are no singular points of the given analytic function within the circle.

Consider now a region S bounded by two concentric circles C_1 C_2, such that $f(z)$ has no singular points in S and that it converges uniformly to finite values along each circle.



Let z_0 be the common center of these circles. We can express the function being considered by means of the Cauchy integral formula.

$f(z)$ in S can then be expressed as

$$f(z) = \frac{1}{2\pi i}\int_{C_1}\frac{f(t)dt}{t - z} + \frac{1}{2\pi i}\int_{C_2}\frac{f(t)dt}{t - z}$$

or by taking the integral along C_2 in a negative direction

$$f(z) = \frac{1}{2\pi i}\int_{C_1}\frac{f(t)dt}{t - z} - \frac{1}{2\pi i}\int_{C_2}\frac{f(t)dt}{t - z}$$

where t is taken in counter clock wise direction along both circles.

Since z is any point of S then for the first integral

$$|z - z_0| < |t - z_0|.$$

This first integral defines a function of z which is holomorphic for all values of z within C_1, and so we can expand this function by means of Taylor's expansion.

Let it be represented by

$$\phi(z) = a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + - - - - - - - a_n(z - z_0)^n + - -$$

where $$a_n = \frac{1}{2\pi i}\int_{C_1} \frac{f(t)dt}{(t-z_0)^{n+1}} \;;\; |z - z_0| < |t - z_0| \text{ , and n is 0, 1, 2,}$$ - - - - - .

The second integral defines a function holomorphic for values exterior to C_2 that is where $|z - z_0| > |t - t_0|$ and t being given values along C_2.

To obtain the expansion of this function we take the function $\frac{1}{t - z}$ and expand it as follows:

$$\frac{1}{t - z} = \frac{1}{z - z_0}\left(\frac{z - z_0}{t - z}\right) = \frac{-1}{z - z_0}\left(\frac{1}{1 - \frac{t - z_0}{z - z_0}}\right)$$

$$= \frac{-1}{z - z_0} - \frac{t - z_0}{(z - z_0)^2} - \frac{(t - z_0)^2}{(z - z_0)^3} - - - - - - - \frac{(t - z_0)^{n-1}}{(z - z_0)^n} - \cdots$$

This taken as a series in t, where $|z - z_0| > |t - z_0|$, converges uniformly for any constant value of z. Values of z outside of C_2 therefore make it convergent.

Multiplying by f(t) gives

$$\frac{f(t)}{t - z} = \frac{-f(t)}{(z - z_0)} - \frac{(t - z_0)f(t)}{(z - z_0)^2} \text{ etc.}$$

which may be integrated term by term as it is uniformly convergent.

$$\therefore \psi(z) = -\frac{1}{2\pi i}\int_{C_2} \frac{f(t)dt}{t - z}$$

$$= \frac{1}{2\pi i}\left\{\frac{1}{z - z_0}\int_{C_2} f(t)dt + \frac{1}{(z - z_0)^2}\int_{C_2}(t-z_0)f(t)dt + - - -\right.$$

$$\left. + \frac{1}{(z - z_0)^n}\int_{C_2}(t - z_0)^{n-1} f(t)dt + - - - - - - -\right.$$

$$\therefore \psi(z) = a_{-1}(z - z_0)^{-1} + a_{-2}(z - z_0)^{-2} + - - - - - a_{-n}(z - z_0)^{-n} + - - -$$

where the coefficient a_{-1}, a_{-2}, etc. are the integrals determined above x $\frac{1}{2\pi i}$.

Consequently for values in S the function f(z) can be written as the sum of $\phi(z)$ and $\psi(z)$ and since the two circles can be deformed into one curve C wholly in S without passing over a singular point of the integrand the coefficients of the two

series may be expressed in terms of the integrals taken over the curve C.

We have then that if $f(z)$ is holomorphic in an annular region S bounded by two concentric circles about a given point, z, then within this region $f(z)$ can be represented by a series of the form

$$\sum_{-\infty}^{\infty} a_n (z - z_0)^n \quad \text{where}$$

$$a_n = \frac{1}{2\pi i} \int_C \frac{f(t)dt}{(t - z_0)^{n+1}}$$ and C is any ordinary curve lying wholly within S and enclosing the inner circle. This Series is called Laurent's Series.

By means of this theorem, taking C_2 infinitely small but not 0, and a transformation $z = \frac{1}{z'}$, we can show that if $z = \infty$ is a pole of order k of a given function $f(z)$ then in the neighborhood of $z = \infty$ the expansion of $f(z)$ is of the form

$$f(z) = a_{-k} z^k + a_{-k+1} z^{k-1} + - - - - - a_{-1} z + a_0 + \frac{a_1}{z} + - - - - - \frac{a_n}{z^n}$$

or our function can be

$$f(z) = a_n z^n + a_{n-1} z^{n-1} + - - - - - - - a_1 z + F(z)$$

where $F(z)$ has $z = \infty$ as a regular point.

Since $f(z)$ is holomorphic everywhere in the finite plane this same expansion holds for all finite values of z and $F(z)$ must be holomorphic in the finite as well as the infinite portion of the plane and therefore must be a constant, or

$$f(z) = a_n z^n + a_{n-1} z^{n-1} + - - - - - - a_1 z + a_0$$

and is a rational integral function.

VI.

REGULARITY OF $f(z)$. The expression "A function $f(z)$ has such and such a property at infinity" means that $\phi(z') = f\left(\frac{1}{z'}\right)$, considered as a function of z', has this property in the neighborhood of the point $z' = 0$.

When a function $f(z)$ of a complex variable is regular in the neighborhood of a point $z = 0$, the point itself excluded; and when further an integer n can be found

such that the product $z^n \cdot f(z) = f_1(z)$ can be made a function regular at $z = 0$ by assigning to it at $z = 0$ a definite finite value different from zero, then we say that $z = 0$ is a pole of $f(z)$ of order n.

If we assign the value zero to the reciprocal function $\frac{1}{f(z)}$ at the point $z = 0$, there is defined in this way a function regular in a certain neighborhood about the point $z = 0$, this point itself included.

We can assign a neighborhood about the point $z = 0$ in which $f_1(z)$ is everywhere different from zero and therefore $\frac{1}{f_1(z)}$ is regular so $\frac{1}{f(z)} = z^n . \frac{1}{f_1(z)}$ is regular there.

$f_1(z)$ can be developed in a series of the form $f_1(z) = a_0 + a_1 z + - - + a_n z^n + - -$ and so

A function $f(z)$, which has a pole of order n at $z = 0$, has a development in this neighborhood of the form

$$f(z) = a_0 z^{-n} + a_1 z^{-n+1} + - - - a_{n-1} z^{-1} + a_n + a_{n+1} z + - - - -$$

Considering $f(z) = f\left(\frac{1}{z'}\right) = \phi(z')$ we can investigate the behavior of a function $f(z)$ at infinity.

For $z' = 0$ this does not define the symbol $\phi(z')$. When it is possible by the preliminary statements to make $\phi(z')$ regular in the neighborhood of the origin we say <u>f(z) is regular at infinity.</u>

If a function of a complex argument is regular in a circle about the origin, it can then be developed, for all points (z) within this circle, in a convergent series of powers of z with positive, integral increasing exponents.

VII

Using $\phi(z')$ and then transforming to z,

A function regular at infinity can be developed in a series;

$$f(z) = a_0 + a_1 z^{-1} + a_2 z^{-2} + - - - - - - a_n z^{-n} + - - -$$

of powers of z with negative, integral, decreasing powers, which converges absolutely

outside of a certain circle with $z = 0$ as a center. Conversely, such a series always represents a function regular at infinity.

VIII

We can also obtain (See page 8)

If a function has a pole of n^{th} order at infinity, it can be developed in a series of the form

$$f(z) = a_{-n}z^{n} + a_{-n+1}z^{n-1} + - - - a_{-2}z^{2} + a_{-1}z + a_0 + a_1 z^{-1} + a_2 z^{-2} + a_n z^{-n} + - - + - -$$

IX

From the above it is possible to show that a function, which is regular everywhere except at infinity and has an n- fold pole at infinity, is a rational integral function of the n^{th} degree.

We have shown that such a function can be developed into a form

$$f(z) = a_{-n}z^{n} + a_{-n+1}z^{n-1} + - - a_{-2}z^{2} + a_{-1}z + a_0 + a_1 z^{-1} + a_2 z^{-2} + - - - a_n z^{-n} + - - - \text{ at infinity.}$$

If we take $\psi(z) = a_{-n}z^{n} + a_{-n+1}z^{n-1} + - - a_{-1}z + a_0$

and form $f(z) - \psi(z) = a_1 z^{-1} + a_2 z^{-2} + - - - a_n z^{-n} + - - - - - -$

this function will be regular everywhere except at infinity for $f(z)$ is regular there and $\psi(z)$ is also regular there, i. e. everywhere except at infinity, on account of its form.

But from its form and previous discussion it is regular at infinity and therefore constant. Its value for $z = \infty$ is 0 and therefore everywhere $= 0$.

$\therefore f(z) =$ the rational integral function $\psi(z)$.

X.

We will show that

A function $f(z)$ which is regular everywhere over the whole plane with the exception of a finite number of poles is a rational function.

Let a_v $(v = 1, 2, 3, --- n)$ be the poles of $f(z)$ on the finite part of the plane, k_v, their order; let

$$\psi_v(z) = \sum_{m=1}^{k_v} \frac{a_v \cdot m}{(z-a_v)^m}$$

be the terms with negative exponents in the development in a series valid for the neighborhood of a_v.

Forming the rational function $\psi(z) = \sum_{v=1}^{m} \psi_v(z)$ the difference $f(z) - \psi(z)$ is regular everywhere except at infinity. At infinity it has a pole or is regular determined by the condition of $f(z)$.

It is either a rational integral function by the previous theorem or a constant as proved in section IX.

Therefore $f(z)$ is equal to the sum of $\psi(z)$ and say $r(z)$ the above function. This sum is a rational function.

Proofs of the Fundamental Theorem of Algebra

Proof #1

This is a proof that every rational integral equation has at least one root, and is based on the fact that no value of $f(z)$ can be its minimum value unless the value of z used makes $f(z)$ vanish.

Given the rational integral function

$f(z) = a_0 z^n + a_1 z^{n-1} + ---- a_n$; a value of z exists for which $f(z)$ vanishes.

In $f(z)$ set $z = x + iy$, where x and y are real, and having expanded

$a_0(x + iy)^n$, $a_1(x + iy)^{n-1}$, - - - - - by the binomial theorem collect all the real terms in the results, and likewise all the imaginary terms. The form of $f(z)$ may then become

$f(z) = \phi(x,y) + i\,\psi(x,y)$ where $\phi(x,y)$ and $\psi(x,y)$ denote real polynomials in x, y, and therefore have

$$|f(z)| = \left[|\phi(x,y)|^2 + |\psi(x,y)|^2\right]^{\frac{1}{2}}$$

We can now find a positive number such as C, so that the roots of $f(z) = 0$ (if there be any) are less, numerically, than C.

If $C' = \frac{C}{\sqrt{2}}$ evidently $|z|$ or $(x^2 + y^2)^{\frac{1}{2}}$ is less than C for all values of C such that $-C' < x < C'$, $-C' < y < C'$.

But in this number region $(-C', C', -C', C')$ which is a rectangle and includes its boundaries, $\left[\phi(x,y)^2 + \psi(x,y)^2\right]^{\frac{1}{2}}$ is a continuous function of x and y. It therefore has a minimum value in this closed region, say when $x = x_0$, $y = y_0$.

If $z_0 = x_0 + iy_0$ then $|f(z_0)| = \left[\phi(x_0y_0)^2 + \psi(x_0y_0)^2\right]^{\frac{1}{2}} = 0$

For since $|f(z_0)|$ is the minimum value of $f(z)$ we can not make $|f(z)| < |f(z_0)|$.

Therefore $|f(z_0)| = 0$ since otherwise by Theorem I Section I we would choose z so that $|f(z)| < |f(z_0)|$.

Hence $|f(z)|$ and therefore $f(z)$, vanishes when $z = z_0$, that is z_0 is a root of the equation $f(z) = 0$.

The form we used was a rational integral function of z.

$\therefore$ The rational integral equation $f(z) = 0$ has at least one root.

We can readily extend this proof by elementary methods to show that $f(z) = 0$ has n roots.

Proof #2

Sometimes we find the statement of the fundamental theorem given in the form, Every rational integral equation of the n^{th} degree has n roots. This is the case in a proof given by Burnside and Panton in their Theory of Equations.

Let $f(z) = a_0 z^n + a_1 z^{n-1} + a_2 z^{n-2} + - - - - a_{n-1} z + a_n$ be a rational integral function of z.

Suppose only that it can not vanish for any infinite value of z.

Let z describe a circle so large that no root of f(z) exists outside.

If $f(z) = z^n(a_0 + a_1 z' + a_2 z'^2 + - a_n z'^n)$

$= z^n \phi(z')$ where $z' = \frac{1}{z}$

z' whose modulus is the reciprocal of the modulus of z, will describe a small circle containing a portion of the plane corresponding to the part of the plane outside the circle described by z, and no root of $\phi(z') = 0$ will be inside this circle.

Hence corresponding to the description of the whole circle by z, the variation of amplitude of $\phi(z')$ will be 0.

$\therefore$ since variation of amplitude of f(z) equals variation of amplitude of z^n + variation of amplitude of $\phi(z')$, the variation of amplitude of f(z) = variation of amplitude of z^n.

If $z = r(\cos\theta + i \sin\theta)$, $z^n = r^n(\cos n\theta + i \sin n\theta)$ the increment of amplitude of $\theta = 2\pi$. $\therefore$ variation of amplitude of z^n or $f(z) = 2\pi n$.

From the above proof (See Section II) this is 2π times the number of roots. Hence n or the order of the function gives the number of roots of f(z).

Proof #3

We will now consider a proof which in itself is perhaps the simplest of all the proofs. It proves that every polynomial of the n^{th} degree has at least one root if $n > 0$.

By use of Section IV, and taking f(z) any polynomial of degree greater than 0,

Suppose $f(z) \neq 0$ for any value of z.

Then $\frac{1}{f(z)}$ is always finite and it is analytic everywhere. Hence it is a constant.

It can not be a constant if as we suppose $n > 0$, hence our supposition that $f(z) \neq 0$ for any z is wrong of $f(z)$ must $= 0$ for some z and hence has at least one root.

Proof #4

This proof depends on development of functions in series and a consideration of singular points both essential and non-essential.

Theorem. If $f(z)$ is a rational integral function; then the equation $f(z) = a_0 + a_1 z + a_2 z^2 + - - - - - - - a_n z^n = 0$ has at least one root.

$f(z)$ being a rational integral function indicates that it is a polynomial of the above type where a_0, a_1, etc. are constants and n is a positive integer. If we also consider $a_n \neq 0$ then the function is of the n^{th} degree.

Every function of such type is a single valued function, that is, for any assigned value of z, $f(z)$ has one value only.

In order that a single valued function of this type shall be a rational integral function it is necessary and sufficient that it have no singular points in the finite portion of the complex plane and that it have a pole at infinity.

The fact that it is holomorphic for all finite values of z shows that it has no singular point in the finite region of the plane.

At $z = \infty$; if we put $z = \frac{1}{z'}$, and call

$$\phi(z') = a + \frac{a_1}{z'} + \frac{a_2}{z'^2} + - - - \frac{a_n}{z'^n}$$

This function has a pole of order n at $z' = 0$ and hence $f(z)$ has a pole of the same order at $z = \infty$.

To show that these conditions are also sufficient, assume that $f(z)$ has no singular points in the finite region and that it has a pole of any order n, at infinity.

To develop such a function for values of z in the neighborhood of infinity we must use Laurent's expansion which is developed in Section V.

Putting $z = \frac{1}{z'}$, in the rational integral function $f(z)$, it follows that $\phi(z') = f\left(\frac{1}{z'}\right)$ has a pole at $z' = 0$.

If $z = z_0$ is a pole of order k of the analytic function $f(z)$, then $\frac{1}{f(z)}$ is

holomorphic in the neighborhood of z_0 and has a zero point of order k at z_0 and conversely.

Therefore $\frac{1}{\phi(z')}$ is holomorphic in the neighborhood of the origin.

$\therefore \frac{1}{f(z)}$ must be holomorphic in the neighborhood of $z = \infty$.

But every analytic function which is not a constant must have at least one singular point either in the finite portion of the plane or at infinity.

Since $\frac{1}{f(z)}$ can not have a singularity at infinity there must be at least one singular point in the finite region.

An essential singularity in $\frac{1}{f(z)}$ in the finite region would require a singularity in f(z) in the finite region. Such a singularity can not exist for f(z) is holomorphic in the finite region.

This requires the singularity $\frac{1}{f(z)}$ to be removable.

Hence z must be a pole of $\frac{1}{f(z)}$ or in other words a zero of f(z).

This proves our theorem that f(z) = 0 must have at least one root.

We can extend the theorem to include the fact that there are n roots.

By successively removing its roots we have each time a rational integral function remaining of one less power than the previous.

By our theorem this also has a root.

When we have removed n-1 of these roots we will have a first degree rational integral equation which has one root. $\therefore$ our equation has n roots.

Proof #5

Taking a rational integral function of the m^{th} degree g(z) and applying the results of Section X to its reciprocal, $\frac{1}{g(z)} = \Psi(z) + r(z)$ where r(z) must be a constant as $\frac{1}{g(z)}$ is regular at infinity.

Reducing to a common denominator $\frac{1}{g(z)} = \frac{h_1(z)}{h_2(z)}$ where $h_1(z)$ is at most of equal degree with $h_2(z)$ which we will call of k degree.

$$h_2(z) = g(z) \cdot h_1(z)$$

From this equation, $m \gtreqless k$.

By hypothesis $\frac{1}{g(z)}$ has a finite number of poles. The number of zeros that $g(z)$ has is the same as the poles of $\frac{1}{g(z)}$. These are, by $m \gtreqless k$, at least m.

But by elementary methods which involve processes valid for complex numbers it can be shown that an equation of the n^{th} degree has no more than n roots unless it is an identity (all coefficients equal to zero).

$\therefore$ every algebraic equation of the n^{th} degree has exactly n roots in the field of complex numbers of the form $a + bi$, where multiple roots are counted according to their order of multiplicity.

Proof #6

The last proof I will present is, of all, the most easily understood.

Let $F(z)$ and $\phi(z)$ be two functions analytic in the interior of the closed curve C, continuous on the curve itself, and such that on the entire curve C, $|\phi(z)| < |F(z)|$.

Then the equations $F(z) = 0$, $F(z) + \phi(z) = 0$ will have the same number of roots in the interior of C, for

$$F(z) + \phi(z) = F(z)\left[1 + \frac{\phi(z)}{F(z)}\right]$$ and

as point z describes the boundary C, the point $Z = 1 + \frac{\phi(z)}{F(z)}$ describes a closed curve lying entirely within the circle of unit radius about the point $Z = 1$ as a center, since $|Z - 1| < 1$ along the entire curve C. Hence the angle of that factor returns to its initial value after the variable z has described the boundary C, and the variation of the angle of $F(z) + \phi(z)$ is equal to the variation of the angle of $F(z)$.

Now let $f(z)$ be a polynomial of the m^{th} degree with any coefficients whatever.

Let $F(z) = A_0 z^m$

and $\phi(z) = A_1 z^{m-1} + \text{---------} A_m$

$f(z)$ then $= F(z) + \phi(z)$

Choose a positive number R so large that

$$\left|\frac{A_1}{A_0}\right| \frac{1}{R} + \left|\frac{A_2}{A_0}\right| \frac{1}{R^2} + \text{ - - } + \left|\frac{A_m}{A_0}\right| \frac{1}{R^m} < 1$$

Then along the entire circle C with origin as center and radius greater than R

$$\frac{\phi(z)}{F(z)} < 1$$

Hence by the proof above $f(z)$ will have the same number of roots within the circle as the equation $F(z) = 0$.

But $F(z) = A_0 z^m = 0$ has m roots.

$\therefore$ $f(z)$ has m roots within circle C.

If C is taken large enough it can include all the finite region.

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